Available at www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®
ELSEVIER
Journal of Approximation Theory 125 (2003) 74–84
JOURNAL OF Approximation Theory
http://www.elsevier.com/locate/jat

On strengthened Hardy and Pólya–Knopp's inequalities

Aleksandra Čižmešija,[a] Josip Pečarić,[b] and Lars–Erik Persson[c,*]

[a] *Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia*
[b] *Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia*
[c] *Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden*

Received 11 December 2002; accepted in revised form 16 September 2003

Communicated by Arno B.J. Kuijlaars

Abstract

In this paper we prove a strengthened general inequality of the Hardy–Knopp type and also derive its dual inequality. Furthermore, we apply the obtained results to unify the strengthened classical Hardy and Pólya–Knopp's inequalities deriving them as special cases of the obtained general relations. We discuss Pólya–Knopp's inequality, compare it with Levin–Cochran–Lee's inequalities and point out that these results are mutually equivalent. Finally, we also point out a reversed Pólya–Knopp type inequality.

MSC: 26D15

Keywords: Inequalities; Weights; Hardy's inequality; Pólya–Knopp's inequality; Levin–Cochran–Lee's inequality; Reversed Pólya–Knopp-type inequality

1. Introduction

In paper [7] Hardy announced, and then proved in [8], a highly important classical integral inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \tag{1}$$

*Corresponding author. Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden. Fax: +46-920-91073.

E-mail addresses: cizmesij@math.hr (A. Čižmešija), pecaric@hazu.hr (J. Pečarić), larserik@sm.luth.se (L.–E. Persson).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2003.09.007

the so-called Hardy's inequality, where $p>1$ and $f \in L^p(0,\infty)$ is a non-negative function. On the other hand, the following related exponential integral inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) dx < e \int_0^\infty f(x)\,dx \tag{2}$$

holds for positive functions $f \in L^1(0,\infty)$. This well-known inequality is many times referred to as Knopp's inequality, with the reference to paper [11]. However, inequality (2) was certainly known before the mentioned Knopp's paper and Hardy himself (see [8, p. 156]) claimed that it was G. Pólya who pointed it out to him earlier (probably by using the limit argument below). Therefore, we prefer to call (2) by name Pólya–Knopp's inequality (see also [12,13]). Note that the discrete version of (2) is surely due to Carleman [1].

It is important to note that inequalities (1) and (2) are closely related since (2) can be obtained from (1) by rewriting it with the function f replaced by $f^{1/p}$ and letting $p\to\infty$. Therefore, Pólya–Knopp's inequality may be considered as a limiting relation of Hardy's inequality. Moreover, the constant factors appearing on the right-hand sides of both inequalities (1) and (2) are sharp (the best possible), that is, they cannot be replaced by any smaller constants. For further remarks concerning the history and properties of inequalities (1) and (2) and their generalizations see e.g. [9] or [15], and also [12,13].

Since Hardy and Pólya discovered inequalities (1) and (2), they have been discussed by several authors, who either reproved them using various techniques, or applied and generalized them in many different ways. Here, we just emphasize monographs [9,13,15,16], related to this topic, and mention Refs. [2–6,10,14,18,19], all of which to some extent have guided us in the research we present here.

In particular, following the results of Yang et al. from [18,19], in paper [4] Čižmešija and Pečarić obtained the so-called strengthened Hardy and Pólya–Knopp-type inequalities. These are relations of the same type as (1) and (2) but with two differences: (i) the outer integrals on their both sides are, instead over $(0,\infty)$, taken over $(0,b)$ or (b,∞), where $b\in\mathbb{R}$, $b>0$, is arbitrary; (ii) the function under the sign of integration on the right-hand side is multiplied by a certain function $0\leqslant v(x)\leqslant 1$ (see Corollaries 1 and 2 below). These sharp inequalities were derived by using a technique of mixed-means inequalities, introduced in [2]. Later on, in [5], similar results were proved in another way, using a different method.

On the other hand, in [10] Kaijser et al. pointed out that both (1) and (2) are just special cases of the much more general Hardy–Knopp-type inequality for positive functions f,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x}, \tag{3}$$

where Φ is a convex function on $(0,\infty)$. This shows that both Hardy and Pólya–Knopp's inequality can be derived by using only convexity, and gives an elegant new proof of these inequalities.

Our aim in this paper is to merge and generalize the recent results from [4,5,10]. First, we generalize relation (3) by adding weight functions and truncating the range of integration to $(0, b)$. In fact, we obtain a strengthened inequality of the Hardy–Knopp type and also prove the so-called dual inequality to this relation, that is, an inequality with the outer integrals taken over (b, ∞) and with the inner integral on the left-hand side taken over (x, ∞). Thus we unify the strengthened Hardy and Pólya–Knopp's inequalities and derive them as special cases of our more general results. Finally, we discuss the Pólya–Knopp's inequality, compare it to the Levin–Cochran–Lee's inequalities from [6,14] (cf. also [3,4,15]), and point out that these results are mutually equivalent.

Conventions. Throughout this paper, all functions are assumed to be measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight function u we mean a non-negative measurable function on the actual interval. Motivated by the well-established concept of duality related to weighted Lebesgue spaces and by the fact that the Hardy operators

$$(Hf)(x) = \int_0^x f(t)\,dt \quad \text{and} \quad (\tilde{H}f)(x) = \int_x^\infty f(t)\,dt, \quad x > 0$$

are mutually conjugate, that is, $H^* = \tilde{H}$ (for further information see e.g. [13, Chapter 1; 16, Chapter 1]), we shall call an inequality with the inner integral taken over (x, ∞) on its left-hand side to be dual to the one of the same type, with $\int_x^\infty$ replaced by $\int_0^x$.

2. The main results

We state and prove a strengthened Hardy–Knopp-type inequality that generalizes inequality (3). It is given in the following theorem.

Theorem 1. *Suppose $0 < b \leqslant \infty$, $u : (0, b) \to \mathbb{R}$ is a non-negative function such that the function $x \mapsto \frac{u(x)}{x^2}$ is locally integrable in $(0, b)$, and the function v is defined by*

$$v(t) = t \int_t^b \frac{u(x)}{x^2}\,dx, \quad t \in (0, b).$$

If the real-valued function Φ is convex on (a, c), where $-\infty \leqslant a < c \leqslant \infty$, then the inequality

$$\int_0^b u(x)\Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^b v(x)\Phi(f(x))\frac{dx}{x} \tag{4}$$

holds for all integrable functions $f : (0, b) \to \mathbb{R}$, such that $f(x) \in (a, c)$ for all $x \in (0, b)$.

Proof. Let $f:(0,b)\to\mathbb{R}$ be an arbitrary integrable function with values in (a,c). Applying Jensen's inequality and Fubini's theorem we obtain

$$\begin{aligned}\int_0^b u(x)\Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} &\leqslant \int_0^b u(x)\left(\int_0^x \Phi(f(t))\,dt\right)\frac{dx}{x^2}\\ &= \int_0^b \Phi(f(t))\int_t^b u(x)\frac{dx}{x^2}\,dt\\ &= \int_0^b v(t)\Phi(f(t))\frac{dt}{t}\end{aligned}$$

and the proof is complete. □

Remark 1. Especially, if the weight function u is chosen to be $u(x)\equiv 1$, then in Theorem 1 we have

$$v(x)=\begin{cases} x\displaystyle\int_x^b \frac{dt}{t^2}=1-\frac{x}{b}, & b<\infty,\\ 1, & b=\infty,\end{cases}\tag{5}$$

so in the case when $b<\infty$ inequality (4) reads

$$\int_0^b \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x}\leqslant\int_0^b\left(1-\frac{x}{b}\right)\Phi(f(x))\frac{dx}{x},$$

while for $b=\infty$ it becomes (3). Therefore, relation (4) may be considered as a generalization of (3), that is, as a new Hardy–Knopp-type inequality.

Remark 2. Note that if the function Φ in Theorem 1 is concave, then (4) holds with the reversed sign of inequality.

Our analysis will be continued by formulating and proving a dual result to Theorem 1.

Theorem 2. *For $0\leqslant b<\infty$, let $u:(b,\infty)\to\mathbb{R}$ be a non-negative locally integrable function in (b,∞) and the function v be given by*

$$v(t)=\frac{1}{t}\int_b^t u(x)\,dx,\quad t\in(b,\infty).$$

If the real-valued function Φ is convex on (a,c), where $-\infty\leqslant a<c\leqslant\infty$, then the inequality

$$\int_b^\infty u(x)\Phi\left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)\frac{dx}{x}\leqslant\int_b^\infty v(x)\Phi(f(x))\frac{dx}{x}\tag{6}$$

holds for all integrable functions $f:(b,\infty)\to\mathbb{R}$, such that $f(x)\in(a,c)$ for all $x\in(b,\infty)$.

Proof. If $f:(b,\infty)\to\mathbb{R}$ is as in the statement of this theorem, then Jensen's inequality and Fubini's theorem yield

$$\begin{aligned}\int_b^\infty u(x)\Phi\left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)\frac{dx}{x} &\leqslant \int_b^\infty u(x)\left(\int_x^\infty \Phi(f(t))\frac{dt}{t^2}\right)dx\\ &= \int_b^\infty \Phi(f(t))\int_b^t u(x)\,dx\frac{dt}{t^2}\\ &= \int_b^\infty v(t)\Phi(f(t))\frac{dt}{t},\end{aligned}$$

so the proof is complete. $\square$

Remark 3. As in Theorem 1, putting $u(x)\equiv 1$ yields

$$v(x)=\frac{1}{x}\int_b^x dt = 1-\frac{b}{x}. \tag{7}$$

Therefore, relation (6) in this setting can be written in the form

$$\int_b^\infty \Phi\left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)\frac{dx}{x}\leqslant \int_b^\infty \left(1-\frac{b}{x}\right)\Phi(f(x))\frac{dx}{x}.$$

In fact, this inequality may be seen as a dual relation to (3).

Remark 4. Theorem 2 holds also with a concave function Φ, except in that case the sign of inequality in relation (6) is reversed.

3. Applications

Although elementary, the idea presented in the previous section seems to be fruitful. To illustrate this fact, we give some applications of Theorems 1 and 2. Namely, we consider the strengthened classical Hardy and Pólya–Knopp's inequalities and show that they are just special cases of the results mentioned. Thus we have merged these well-known inequalities, and have provided them with new proofs.

First, consider the strengthened Hardy's integral inequality.

Corollary 1. *Let $p,k,b\in\mathbb{R}$ be such that $p>1$, $k\neq 1$ and $b>0$, and let f be a nontrivial and non-negative function.*

(i) *If $k>1$ and $0<\int_0^b x^{p-k}f^p(x)\,dx<\infty$, then*

$$\int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx < \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k}f^p(x)\,dx. \tag{8}$$

(ii) *If* $k<1$ *and* $0<\int_b^\infty x^{p-k}f^p(x)\,dx<\infty$, *then*

$$\int_b^\infty x^{-k}\left(\int_x^\infty f(t)\,dt\right)^p dx<\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k}f^p(x)\,dx. \quad (9)$$

The constant $(p/|k-1|)^p$ *is the best possible for both inequalities.*

Proof. The proof follows from Theorems 1 and 2 by choosing the convex function $\Phi(x)=x^p$ and the weight function $u(x)\equiv 1$.

Consider the case when $k>1$ first. Observing that the weight function v in Theorem 1 is then defined by (5), relation (4) reads

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p \frac{dx}{x} \leqslant \int_0^b \left(1-\frac{x}{b}\right) f^p(x)\,\frac{dx}{x}. \quad (10)$$

Now, replace the parameter b by $a=b^{(k-1)/p}$ and choose for f the function $x\mapsto f(x^{p/(k-1)})x^{p/(k-1)-1}$. Then, with the substitutions $s=t^{p/(k-1)}$ and $y=x^{p/(k-1)}$ respectively, the left-hand side of (10) becomes

$$\begin{aligned}
&\int_0^a \left(\frac{1}{x}\int_0^x f(t^{p/(k-1)})t^{p/(k-1)-1}\,dt\right)^p \frac{dx}{x}\\
&\quad=\left(\frac{k-1}{p}\right)^p \int_0^a \left(\frac{1}{x}\int_0^{x^{p/(k-1)}} f(s)\,ds\right)^p \frac{dx}{x}\\
&\quad=\left(\frac{k-1}{p}\right)^{p+1} \int_0^b y^{-k}\left(\int_0^y f(s)\,ds\right)^p dy.
\end{aligned}$$

Analogously, substituting $y=x^{p/(k-1)}$, on the right-hand side of (10) we obtain

$$\begin{aligned}
&\int_0^a \left(1-\frac{x}{a}\right) f^p(x^{p/(k-1)})x^{p(p/(k-1)-1)}\,\frac{dx}{x}\\
&\quad=\frac{k-1}{p}\int_0^b \left[1-\left(\frac{y}{b}\right)^{\frac{k-1}{p}}\right] y^{p-k}f^p(y)\,dy,
\end{aligned}$$

so relation (8) is proved. Note that the inequality sign in (8) is strict, owing to the conditions on f from the statement of the theorem and to the fact that the function Φ is strictly increasing.

Now, suppose that $k<1$. According to Theorem 2, considered with the weight $u(x)\equiv 1$, we have (7). Thus, relation (6) can be written in the form

$$\int_b^\infty \left(x\int_x^\infty f(t)\,\frac{dt}{t^2}\right)^p \frac{dx}{x} \leqslant \int_b^\infty \left(1-\frac{b}{x}\right) f^p(x)\,\frac{dx}{x}. \quad (11)$$

Replacing the function f in (11) by $x\mapsto f(x^{p/(1-k)})x^{p/(1-k)+1}$, the parameter b by $a=b^{(1-k)/p}$, and making a similar sequence of substitutions as in the previous case, on the left-hand side of (11) we have

$$\begin{aligned}
&\int_a^\infty \left(x \int_x^\infty f(t^{p/(1-k)}) t^{p/(1-k)+1} \frac{dt}{t^2} \right)^p \frac{dx}{x} \\
&\quad = \left(\frac{1-k}{p} \right)^p \int_a^\infty \left(x \int_{x^{p/(1-k)}}^\infty f(s)\, ds \right)^p \frac{dx}{x} \\
&\quad = \left(\frac{1-k}{p} \right)^{p+1} \int_b^\infty y^{-k} \left(\int_y^\infty f(s)\, ds \right)^p dy,
\end{aligned}$$

while the right-hand side of (11) is

$$\begin{aligned}
&\int_a^\infty \left(1 - \frac{a}{x}\right) f^p(x^{p/(1-k)}) x^{p(p/(1-k)+1)} \frac{dx}{x} \\
&\quad = \frac{1-k}{p} \int_b^\infty \left[1 - \left(\frac{b}{y}\right)^{\frac{1-k}{p}} \right] y^{p-k} f^p(y)\, dy.
\end{aligned}$$

Hence, (9) is proved, with the strict sign of the inequality again. The proof that $(p/|k-1|)^p$ is the best possible constant for (8) and (9) is given in [4]. □

Remark 5. Observe that Hardy's inequality written in the form (10), as also its dual inequality (11), hold also for $p = 1$, but this is meaningless if they are written in the form (8) and (9).

Remark 6. Note that by rewriting (8) with $b = \infty$ we obtain the classical Hardy's inequality (in particular, for $k = p$ we have (1)), while its dual inequality is achieved by putting $b = 0$ in (9).

Remark 7. It is easy to see that Corollary 1 holds also for the parameter $p \in (0,1)$, but with a reversed sign of inequality in (8) and (9) since the function $\Phi(x) = x^p$ is in that case concave (see Remarks 2 and 4).

Now, we continue with Pólya–Knopp's inequality and its dual. The results will be stated in a form given in [4,6,14], and after the proof they will be compared with (1).

Corollary 2. *Let $\alpha, \gamma, b \in \mathbb{R}$ be such that $\alpha \neq 0$ and $b > 0$, and let f be a positive function.*
(i) *If $\alpha > 0$ and $0 < \int_0^b x^{\gamma-1} f(x)\, dx < \infty$, then*

$$\begin{aligned}
&\int_0^b x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha} \int_0^x t^{\alpha-1} \log f(t)\, dt \right] dx \\
&\quad < e^{\gamma/\alpha} \int_0^b \left[1 - \left(\frac{x}{b}\right)^\alpha \right] x^{\gamma-1} f(x)\, dx. \qquad (12)
\end{aligned}$$

(ii) *If* $\alpha<0$ *and* $0<\int_b^\infty x^{\gamma-1}f(x)\,dx<\infty$, *then*

$$\int_b^\infty x^{\gamma-1}\exp\left[-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\,dt\right]dx$$
$$<e^{\gamma/\alpha}\int_b^\infty\left[1-\left(\frac{b}{x}\right)^{-\alpha}\right]x^{\gamma-1}f(x)\,dx. \tag{13}$$

The constant $e^{\gamma/\alpha}$ *is the best possible for both inequalities.*

Proof. The proof follows from Theorems 1 and 2 if the functions Φ and u are chosen to be $\Phi(x)=e^x$ and $u(x)\equiv 1$. First, let us prove the case when $\alpha>0$. Since the weight function v from Theorem 1 is given by (5), the inequality (4) in this setting reads

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x}\leqslant\int_0^b\left(1-\frac{x}{b}\right)\exp(f(x))\frac{dx}{x}. \tag{14}$$

Now, replace b by $a=b^\alpha$ and choose for f the function $x\mapsto\log(x^{\gamma/\alpha}f(x^{1/\alpha}))$. Then, with the substitutions $s=t^{1/\alpha}$ and $y=x^{1/\alpha}$, the left-hand side of (14) becomes

$$\begin{aligned}
&\int_0^a \exp\left[\frac{1}{x}\int_0^x \log(t^{\gamma/\alpha}f(t^{1/\alpha}))\,dt\right]\frac{dx}{x}\\
&=\int_0^a \exp\left[\frac{1}{x}\int_0^x \log(t^{\gamma/\alpha})\,dt+\frac{1}{x}\int_0^x \log f(t^{1/\alpha})\,dt\right]\frac{dx}{x}\\
&=e^{-\gamma/\alpha}\int_0^a x^{\gamma/\alpha}\exp\left[\frac{1}{x}\int_0^x \log f(t^{1/\alpha})\,dt\right]\frac{dx}{x}\\
&=e^{-\gamma/\alpha}\int_0^a x^{\gamma/\alpha-1}\exp\left[\frac{\alpha}{x}\int_0^{x^{1/\alpha}} s^{\alpha-1}\log f(s)\,ds\right]dx\\
&=\alpha e^{-\gamma/\alpha}\int_0^b y^{\gamma-1}\exp\left[\frac{\alpha}{y^\alpha}\int_0^y s^{\alpha-1}\log f(s)\,ds\right]dy.
\end{aligned}$$

Similarly, on the right-hand side of (14) we obtain

$$\begin{aligned}
&\int_0^a\left(1-\frac{x}{a}\right)\exp[\log(x^{\gamma/\alpha}f(x^{1/\alpha}))]\frac{dx}{x}\\
&=\int_0^a\left(1-\frac{x}{a}\right)x^{\gamma/\alpha}f(x^{1/\alpha})\frac{dx}{x}=\alpha\int_0^b\left[1-\left(\frac{y}{b}\right)^\alpha\right]y^{\gamma-1}f(y)\,dy,
\end{aligned}$$

so relation (12) is proved. Observe that, under the conditions on f from the statement of the theorem and considering the properties of the function Φ, the sign of the inequality in (12) is strict.

In the case when $\alpha<0$, the inequality (13) will be derived from Theorem 2. Considering (7), relation (6) now has the form

$$\int_b^\infty \exp\left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)\frac{dx}{x}\leqslant\int_b^\infty\left(1-\frac{b}{x}\right)\exp(f(x))\frac{dx}{x}. \tag{15}$$

Thus, replacing b and f by $a = b^{-\alpha}$ and $x \mapsto \log(x^{-\gamma/\alpha} f(x^{-1/\alpha}))$, after the changes of the variables $s = t^{-1/\alpha}$ and $y = x^{-1/\alpha}$, on the left-hand side of (15) we have

$$\begin{aligned}
&\int_a^\infty \exp\left[x \int_x^\infty \log(t^{-\gamma/\alpha} f(t^{-1/\alpha}))\,\frac{dt}{t^2}\right]\frac{dx}{x}\\
&\quad = \int_a^\infty \exp\left(-\frac{\gamma}{\alpha} x \int_x^\infty \log t\,\frac{dt}{t^2}\right)\exp\left(x \int_x^\infty \log f(t^{-1/\alpha})\,\frac{dt}{t^2}\right)\frac{dx}{x}\\
&\quad = e^{-\gamma/\alpha} \int_a^\infty x^{-\gamma/\alpha-1} \exp\left[-\alpha x \int_{x^{-1/\alpha}}^\infty s^{\alpha-1}\log f(s)\,ds\right] dx\\
&\quad = -\alpha e^{-\gamma/\alpha} \int_b^\infty y^{\gamma-1} \exp\left[-\frac{\alpha}{y^\alpha} \int_y^\infty s^{\alpha-1}\log f(s)\,ds\right] dy,
\end{aligned}$$

while, analogously, the right-hand side of (15) becomes

$$\int_a^\infty \left(1-\frac{a}{x}\right) x^{-\gamma/\alpha} f(x^{-1/\alpha})\,\frac{dx}{x} = -\alpha \int_b^\infty \left[1-\left(\frac{b}{y}\right)^{-\alpha}\right] y^{\gamma-1} f(y)\,dy.$$

Therefore, (13) is proved. Note that from the same reasons as in the previous case the inequality sign is strict. The proof that $e^{\gamma/\alpha}$ is the best possible constant for both inequalities (12) and (13) can be found in [4]. $\square$

Remark 8. By letting $b \to \infty$ in (12) we obtain the inequality

$$\int_0^\infty x^{\gamma-1} \exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right) dx < e^{\gamma/\alpha} \int_0^\infty x^{\gamma-1} f(x)\,dx, \tag{16}$$

due to Cochran and Lee [6]. On the other hand, relation (13), rewritten with $b = 0$, reads

$$\int_0^\infty x^{\gamma-1} \exp\left(-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\,dt\right) dx < e^{\gamma/\alpha} \int_0^\infty x^{\gamma-1} f(x)\,dx. \tag{17}$$

This inequality, dual to (16), was proved by Love [14] (cf. also [15] for both results). Because of some reasons explained in [2,3], relations (16) and (17) are called Levin–Cochran–Lee's inequalities.

Remark 9. Note that by choosing the parameters $\alpha = 1$ and $\gamma = 1$ inequality (12) becomes

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) dx < e \int_0^b \left(1-\frac{x}{b}\right) f(x)\,dx, \tag{18}$$

while substituting $\alpha = -1$ and $\gamma = 1$ in (13) we have

$$\int_b^\infty \exp\left(x \int_x^\infty \log f(t)\,\frac{dt}{t^2}\right) dx < \frac{1}{e} \int_b^\infty \left(1-\frac{b}{x}\right) f(x)\,dx. \tag{19}$$

Relations (18) and (19) are the strengthened Pólya–Knopp's and its dual inequality.

Remark 10. It is important to observe that inequalities (12) and (18) are mutually equivalent, as also relations (13) and (19). Indeed, it is evident that (12) implies (18) by using a suitable choice of the parameters. Now, we have to prove the converse. Replacing b by b^α, choosing for f the function $x\mapsto x^{\gamma/\alpha-1}f(x^{1/\alpha})$, and using the same sequence of substitutions as in the proof of Corollary 2, on the left-hand side of (18) we obtain

$$\begin{aligned}&\int_0^{b^\alpha} \exp\left[\frac{1}{x}\int_0^x \log(t^{\gamma/\alpha-1}f(t^{1/\alpha}))\,dt\right]dx\\ &\quad= e^{1-\gamma/\alpha}\int_0^{b^\alpha} x^{\gamma/\alpha-1}\exp\left(\frac{\alpha}{x}\int_0^{x^{1/\alpha}} s^{\alpha-1}\log f(s)\,ds\right)dx\\ &\quad= \alpha e^{1-\gamma/\alpha}\int_0^{b} y^{\gamma-1}\exp\left(\frac{\alpha}{y^\alpha}\int_0^{y} s^{\alpha-1}\log f(s)\,ds\right)dy,\end{aligned}$$

while the right-hand side of (18) becomes

$$e\int_0^{b^\alpha}\left(1-\frac{x}{b^\alpha}\right)x^{\gamma/\alpha-1}f(x^{1/\alpha})\,dx = \alpha e\int_0^b\left[1-\left(\frac{y}{b}\right)^\alpha\right]y^{\gamma-1}f(y)\,dy.$$

Hence, we have (12). The proof that inequalities (13) and (19) are mutually equivalent is similar. Therefore, Levin–Cochran–Lee's inequalities are not more general than Pólya–Knopp-type inequalities.

Remark 11. According to Corollary 2(i), and its proof (see relation (14)), the following sharp Pólya–Knopp-type inequality holds:

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right)\frac{dx}{x} < \int_0^b\left(1-\frac{x}{b}\right)f(x)\frac{dx}{x}. \tag{20}$$

By using Remark 2 with the concave function $\Phi(x)=\log x$, in a similar way we obtain a reversed Pólya–Knopp-type inequality

$$\int_0^b \log\left(\frac{1}{x}\int_0^x \exp f(t)\,dt\right)\frac{dx}{x} > \int_0^b\left(1-\frac{x}{b}\right)f(x)\frac{dx}{x}. \tag{21}$$

Analogously, we can use Corollary 2(ii), and relation (15) together with Remark 4 (considering the same concave function Φ), to obtain dual forms of (20) and (21), that is, the inequalities

$$\int_b^\infty \exp\left(x\int_x^\infty \log f(t)\frac{dt}{t^2}\right)\frac{dx}{x} < \int_b^\infty\left(1-\frac{b}{x}\right)f(x)\frac{dx}{x}$$

and

$$\int_b^\infty \log\left(x\int_x^\infty \exp f(t)\frac{dt}{t^2}\right)\frac{dx}{x} > \int_b^\infty\left(1-\frac{b}{x}\right)f(x)\frac{dx}{x},$$

respectively.

Remark 12. The results presented in this paper can be generalized to special multidimensional versions where the averages are taken over spheres in $\mathbb{R}^n$ and even over more general infinite spherical cones. This fact can be understood by using polar coordinates and the one-dimensional results we just proved, together with the ideas pointed out in [17]. The present authors plan to present the details of this research, as also some other new complementary results and generalizations, in a forthcoming paper.

Acknowledgments

We thank the careful referee for some good suggestions and ideas, which we have used to improve the final version of this paper.

References

[1] T. Carleman, Sur les fonctions quasi-analytiques, Comptes rendus du V^e Congres des Mathematiciens Scandinaves, Helsingfors, 1922, pp. 181–196.
[2] A. Čižmešija, J. Pečarić, Mixed means and Hardy's inequality, Math. Inequal. Appl. 1 (4) (1998) 491–506.
[3] A. Čižmešija, J. Pečarić, Classical Hardy's and Carleman's inequalities and mixed means, in: T.M. Rassias (Ed.), Survey on Classical Inequalities, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000, pp. 27–65.
[4] A. Čižmešija, J. Pečarić, Some new generalisations of inequalities of Hardy and Levin–Cochran–Lee, Bull. Austral. Math. Soc. 63 (1) (2001) 105–113.
[5] A. Čižmešija, J. Pečarić, On Bicheng–Debnath's generalizations of Hardy's integral inequality, Int. J. Math. Math. Sci. 27 (4) (2001) 237–250.
[6] J.A. Cochran, C.-S. Lee, Inequalities related to Hardy's and Heinig's, Math. Proc. Cambridge Philos. Soc. 96 (1984) 1–7.
[7] G.H. Hardy, Note on a theorem of Hilbert, Math. Z. 6 (1920) 314–317.
[8] G.H. Hardy, Notes on some points in the integral calculus (60), Messenger Math. 54 (1925) 150–156.
[9] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, 2nd Edition, Cambridge University Press, Cambridge, 1967.
[10] S. Kaijser, L.-E. Persson, A. Öberg, On Carleman and Knopp's Inequalities, J. Approx. Theory 117 (2002) 140–151.
[11] K. Knopp, Über reihen mit positiven gliedern, J. London Math. Soc. 3 (1928) 205–211.
[12] A. Kufner, L.-E. Persson, The Hardy inequality—about its history and current status, Research Report 6, Dept. of Math., Luleå University of Technology, 2002 (ISSN 1400-4003).
[13] A. Kufner, L.-E. Persson, Weighted Inequalities of Hardy Type, World Scientific Publishing Co., New Jersey, London, Singapore, Hong Kong, 2003.
[14] E.R. Love, Inequalities related to those of Hardy and of Cochran and Lee, Math. Proc. Cambridge Philos. Soc. 99 (1986) 395–408.
[15] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Inequalities Involving Functions and their Integrals and Derivatives, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.
[16] B. Opic, A. Kufner, Hardy-type inequalities, Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, 1990.
[17] G. Sinnamon, One-dimensional Hardy-type inequalities in many dimensions, Proc. Roy. Soc. Edinburgh Sect. A 128 (4) (1998) 833–848.
[18] B. Yang, L. Debnath, Generalizations of Hardy's integral inequalities, Internat. J. Math. Math. Sci. 22 (3) (1999) 535–542.
[19] B. Yang, Z. Zeng, L. Debnath, On new generalizations of Hardy's integral inequality, J. Math. Anal. Appl. 217 (1998) 321–327.